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Linear Project Second Order-Cone Programme

Zhenhua Li[*1], Wenling Zhao[2]
[*1,2] School of Science,Shandong University of Technology, Zibo,255091,China
lzh73407968@163.com

Abstract

In recent years cone constraint optimization problems has been favored by scholars, especially for the second-order cone constraints programming problem,they have carried on the detailed study, had got the corresponding theoretical achievements. In this paper, on the basis of the existing cone constraint optimization problem, we puts forward the definition of projection second-order cone ,we related properties of this cone and the corresponding convexity function, the linear projected second-order cone constraint programme problem, the dual problem, the optimality conditions ,we can converted it into corresponding second-order cone programme problems.

Keywords: linear projected second-order cone; dual cone constraints; Second-order cone programme problem; dual problem ; convexity function

Introduction

In recent years, with the development of science and technology, the application of computer. Nonlinear optimization obtained rapid development and wide application [7,10,12,13,18,23]. Especially the semidefinite programming and the second-order cone programming problem, f. Alizadeth and d. Goldfarb [1] introduced the second-order cone programming problem in detail, research the linear and nonlinear second-order cone constraints programming problem and quadratic programming problem. They have introduce the relationship between the other convex programming and second-order cone constraints problems, using Jordan operator interior-point method analyzed the application of the second-order cone programming problem, and the semidefinite programming.Scholars use different methods to study the the different cone constraints problem[3,5,12,14]. In general, we often use different method resolve cone constraints programming problem.for example penalty function, primal-dual interior point, smooth function approximation method, semismooth Newton's method and quasi-newton method,power iteration method and projection algorithm method and so on[6,8,12,16,17 [4]. The complementarity system occupies an important status in the constrained optimization problems, So research the property of cone is very important in the complementarity system[5,14,22 [3].In this paper, on the basis of the existing cone constraint optimization problem we put forward the definition of projection second-order cone, search the properties of the cone and its dual cone.

In order to give definitions of projected second-order cone, we first solve below function:

Assumption

$P=(p,p\cdots p)$ and $P\in R^m$, $p\geq 0$, For the function

minimum $f(P)=\frac{1}{2}(x_1-p)^2+\frac{1}{2}(x_2-p)^2+\cdots+\frac{1}{2}(x_m-p)^2$

with $x\in R^m, \sum_{i=1}^{m}x_i\geq 0$.We have $p=\frac{\sum_{i=1}^{m}x_i}{m}$ is the solution of the above function.

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Define 1.1 The $K=\left\{x\left|\frac{(x_1-p)^2+(x_2-p)^2+\cdots+(x_m-p)^2}{\|x\|^2}\le\frac{1}{2}\right.\right\}$ is the project second-order cone. If $x=0$,we assumption $K=\{0\}$.

We be represent project second-order cone as $K=\left\{x\left|m\|x\|^2\le 2\left(\sum_{i=1}^m x_i\right)^2\right.\right\}$ （$\sum_{i=1}^m x_i\ge 0$，$p=\frac{1}{m}\sum_{i=1}^m x_i$ （$\|\cdot\|$ denote euclidean norm）.

Theoren 1.1： K is self dual, And $K=K^+$,so we have $K^+=\left\{x\left|m\|x\|^2\le 2\left(\sum_{i=1}^m x_i\right)^2\right.\right\}$.

Proof: For $\forall x\in K$, $\forall y\in K$,if $x=0$ or $y=0$, It is obvious $\langle x,y\rangle=0$.

If $x\ne 0$, $y\ne 0$,By the define 1.1 we have

$$\frac{\left(\left(x_1-\frac{1}{m}\sum_{i=1}^m x_i\right)^2+\left(x_2-\frac{1}{m}\sum_{i=1}^m x_i\right)^2+\cdots+\left(x_m-\frac{1}{m}\sum_{i=1}^m x_i\right)^2\right)^{\frac{1}{2}}}{\|x\|}\le\frac{1}{2}$$

$$\frac{\left(\left(y_1-\frac{1}{m}\sum_{i=1}^m y_i\right)^2+\left(y_2-\frac{1}{m}\sum_{i=1}^m y_i\right)^2+\cdots+\left(y_m-\frac{1}{m}\sum_{i=1}^m y_i\right)^2\right)^{\frac{1}{2}}}{\|y\|}\le\frac{1}{2}。$$

so $\left\langle\left(x_1-\frac{1}{m}\sum_{i=1}^m x_i, x_2-\frac{1}{m}\sum_{i=1}^m x_i,\cdots,x_m-\frac{1}{m}\sum_{i=1}^m x_i\right),\left(y_1-\frac{1}{m}\sum_{i=1}^m y_i, y_1-\frac{1}{m}\sum_{i=1}^m y_i,\cdots,y_1-\frac{1}{m}\sum_{i=1}^m y_i\right)\right\rangle \ge -\frac{1}{2}\|x\|\cdot\|y\|$ By the left-hand side above we have

$$\langle x,y\rangle-\frac{1}{m}\sum_{i=1}^m x_i\cdot\sum_{i=1}^m y_i\ge-\frac{1}{2}\|x\|\cdot\|y\|$$

$$\langle x,y\rangle\ge\frac{1}{m}\sum_{i=1}^m x_i\cdot\sum_{i=1}^m y_i-\frac{1}{2}\|x\|\cdot\|y\|$$

As $0<\frac{1}{\sqrt{2}}\|x\|^2\le\frac{1}{\sqrt{m}}\left(\sum_{i=1}^m x_i\right)$, $0<\frac{1}{\sqrt{2}}\|y\|^2\le\frac{1}{\sqrt{m}}\left(\sum_{i=1}^m y_i\right)$.

So $\langle x,y\rangle\ge\frac{1}{m}\sum_{i=1}^m x_i\cdot\sum_{i=1}^m y_i-\frac{1}{2}\|x\|\cdot\|y\|\ge 0$.

Above all,we obtain $K=K^+$.

Define 1.2 K^- is the polar cone of K if $K^-=\{y|\langle x,y\rangle\le 0,\forall x\in K\}$.

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Theoren 1.2 : For $\forall x \in K, \forall y \in K$, $x \neq 0$, $y \neq 0$, if $\langle x, y \rangle = 0$, we obtain $x, y \in K_B$.(K_B denote the cone shell).And if $x \in K_B$, we have $n\|x\|^2 = 2\left(\sum_{i=1}^{m} x_i\right)^2$.

Proof: Assumption $x \notin K_B$ .As $\langle x, y \rangle = 0$, $\langle x, y \rangle = 0$ is continuity function, in the domain of $x + \varepsilon B \subset K(\varepsilon > 0)$, there exist $z \in \{x + \varepsilon B\}(\varepsilon > 0)$ satisfi $\langle z, y \rangle < 0$ contradictory the assumption.

We can be proven $y \in K_B$.

Define 1.3 For $x \in K$, $y \in K^+$ if $\langle x, y \rangle = 0$, we have

$$y = t\left(\frac{1}{m}\sum_{i=1}^{m} x_i - \frac{x_1}{2}, \frac{1}{m}\sum_{i=1}^{m} x_i - \frac{x_2}{2}, \cdots, \frac{1}{m}\sum_{i=1}^{n} x_i - \frac{x_m}{2}\right)$$ （$t > 0$, if $x \in \operatorname{int} K$ we assuption $t = 0$）.

Proof: If $x \in \operatorname{int} K$,and $t = 0$,It is obvious.

If $x \in K_B$,From the Theoren 1.2 we know $y \in K_B$.We only need to prove $m\|y\|^2 = 2\left(\sum_{i=1}^{m} y_i\right)^2$,from the define 1.1 we have

$$m\left(\left(\frac{1}{m}\sum_{i=1}^{m} x_i - \frac{x_1}{2}\right)^2 + \left(\frac{1}{m}\sum_{i=1}^{m} x_i - \frac{x_2}{2}\right)^2 + \cdots + \left(\frac{1}{m}\sum_{i=1}^{m} x_i - \frac{x_m}{2}\right)^2\right) = 2\left(\sum_{i=1}^{m} x_i - \frac{1}{2}\sum_{i=1}^{m} x_i\right)^2$$

To selve this equation we have $m\|x\|^2 = 2\left(\sum_{i=1}^{m} x_i\right)^2$.

Proposition 1.1 K is pointed and convex cone.

Proof: Because $K \cap K^- = 0$,so K is pointed .

Now we proof it convex,for $\forall x, y \in K$, $t \in [0,1]$ .we only proof

$$m\{[tx_1 + (1-t)y_1]^2 + [tx_2 + (1-t)y_2]^2 + \cdots + [tx_m + (1-t)y_m]^2\} \leq 2\left[\sum_{i=1}^{m} tx_i + (1-t)y_i\right]^2$$

This nonequality equal

$$m\sum_{i=1}^{m} x_i y_i \leq 2\sum_{i=1}^{m} x_i \sum_{i=1}^{m} y_i$$

from the define 1.1 we have $m\|x\|^2 \leq 2\left(\sum_{i=1}^{m} x_i\right)^2$ and $m\|y\|^2 \leq 2\left(\sum_{i=1}^{m} y_i\right)^2$.

so $\dfrac{\sum_{i=1}^{m} x_i \sum_{i=1}^{m} y_i}{\|x\| \cdot \|y\|} \leq 1$, K is convex cone.

It very important to research the tangent cone, normal cone and the definition of the second order tangent set for the project second-order cone.J. Frederic and H. Ramirecz C [24] had dicuss first and second order optimality condiition for nonlinear second-order programming.. Before we discuss these cones, we first presents the function of the project second-order cone

If K is a project second-order cone we have:

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$$K = \left\{ x \in R^m \middle| \phi(x) = \sqrt{m}\|x\| - \sqrt{2}\sum_{i=1}^{m} x_i \le 0 \right\}.$$

Proposition 1.2 The function $\phi(x) = \sqrt{m}\phi(x) = \sqrt{m}\|x\| - \sqrt{2}\sum_{i=1}^{m} x_i \le 0$ convex and derivable. **Proof:** We need to proof $\phi(z) \ge \phi(x) + \phi(x)'(z - x)$

as $$\sqrt{m}\|z\| - \sqrt{2}\sum_{i=1}^{m} z_i \ge \sqrt{m}\|x\| - \sqrt{2}\sum_{i=1}^{m} x_i + \left((z_1 - x_1),(z_2 - x_2),\cdots,(z_m - x_m)\right)\begin{pmatrix} \dfrac{\sqrt{m}x_1 - \sqrt{2}\|x\|}{\|x\|} \\ \dfrac{\sqrt{m}x_2 - \sqrt{2}\|x\|}{\|x\|} \\ \vdots \\ \dfrac{\sqrt{m}x_m - \sqrt{2}\|x\|}{\|x\|} \end{pmatrix}$$

By the right-hand side above we have

$$\sqrt{m}\|x\| - \sqrt{2}\sum_{i=1}^{m} x_i + \frac{\sqrt{m}}{\|x\|}\sum_{i=1}^{m} x_i z_i + \sqrt{2}\sum_{i=1}^{m} x_i - \sqrt{m}\|x\| - \sqrt{2}\sum_{i=1}^{m} z_i$$

So $\|z\| \ge \dfrac{\sum_{i=1}^{m} x_i z_i}{\|x\|}$. As we know $\dfrac{\sum_{i=1}^{m} x_i z_i}{\|x\|\|z\|} \le 1$. The proof is completed.

Now we consider project second-order cone programming problem

$$\begin{aligned} &\text{(P)} \quad \min \quad cx \\ &sb \quad Ax \quad = b \\ &\qquad\; x \quad \in K \end{aligned}$$

A is a row full rank matric , K is project second-order cone .
The dual of (P)

$$\begin{aligned} &\text{(D)} \max \quad b^{\mathrm{T}}\pi \\ &sb \quad A^{\mathrm{T}}\pi + y \quad = c \\ &\qquad\; y \quad \in K \end{aligned}$$

We know the dual problem is very important in cone constraints programming, for the second-order cone, ice cream cone, cone, geometry cone constraints constraint programming [15,16,17,22 ,22] have discussed .

Assumptions p^* and d^* is the original problem and solution of dual problem respectively, x^* and (π^*, y^*) the corresponding solution sets. We have

$$p^* - d^* = c^{\mathrm{T}}x^* - b^{\mathrm{T}}\pi^* = x^{*\mathrm{T}}y^* \ge 0$$

It is clear that the solution of original problem is the supremum of the dual problem solution, if $p^* - d^* > 0$ then the original problem and the have the duality gap .

If $x^* \in \operatorname{int} K$ (or (π^*, y^*) $y^* \in \operatorname{int} K$), then x^* (or (π^*, y^*) $y^* \in \operatorname{int} K$) is strictly feasible solution of original problem (strictly feasible solution of dual problem).

If $p^* = -\infty$ (or $d^* = +\infty$), then we call the original problem (or the dual problem)is unbounded. If $p^* = +\infty$ (or $d^* = -\infty$), then the original problem (or the dual problem) is no feasible solution. When the original problem and the dual problem has strictly feasible solution, then the duality gap is 0.

For linear projection of second-order cone constraints problem can be converted into into a linear second-order cone constraints.

Let $x_0 = \sqrt{\frac{2}{n}}(x_1 + x_2 + \cdots + x_m)$,we have

$$\begin{aligned} \min \quad & \bar{c}^{\mathrm{T}} \bar{x} \\ sb \quad & \bar{A}\bar{x} = \bar{b} \\ & \bar{x} \in \bar{K} \end{aligned}$$

And $\bar{c} = (0, c_1, c_2, \cdots, c_m)$, $\bar{x} = (x_0, x_1, x_2, \cdots, x_m)$, $\bar{A} = \begin{bmatrix} -\frac{\sqrt{2n}}{2} & -1 & \cdots & -1 \\ 0 & a_{11} & \cdots & a_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{m1} & \cdots & a_{mm} \end{bmatrix}$.

$\bar{b} = (0, b_1, b_2, \cdots, b_m)$. $\bar{K}$ is second-order cone.

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